

Gravitational microlensing in modified gravity theories: Inverse-square theorem

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Microlensing studies are usually based on the lens equation that is valid only to the first order in the gravitational constant G and lens mass M . We consider corrections to the conventional lens equation in terms of differentiable functions, so that they can express not only the second-order effects of GM in general relativity but also modified gravity theories. As a generalization of Ebina et al. (Prog. Theor. Phys. **104** (2000) 1317), we show that, provided that the spacetime is static, spherically symmetric and asymptotically flat, the total amplification by microlensing remains unchanged at the linear order of the correction to the deflection angle, if and only if the correction takes a particular form as the inverse square of the impact parameter, whereas the magnification factor for each image is corrected. It is concluded that the light curve shape by microlensing is inevitably changed and will thus allow us to probe modified gravity, unless a modification to the deflection angle takes the particular form. No systematic deviation in microlensing observations has been reported. For instance, therefore, the Yukawa-type correction is constrained as the characteristic length $> 10^{14}$ m.

§1. Introduction

Since the pioneering work by Paczynski¹⁾ and the subsequent detections of MA-CHOs,^{2),3)} microlensing has been one of the most vital areas in astrophysics. Now, it plays an important role also in searching extra-solar planets.^{4),5)} Light rays are influenced by a local curvature of a spacetime.⁶⁾ It seems natural that the bending angle of light rays in modified gravity theories is different from that in general relativity. Does such a modified bending angle really makes a change in microlensing? We shall examine this problem in this paper.

Conventional studies of microlensing are usually based on the lens equation that maps the source direction into the position on the lens plane, namely the direction of each image due to the lens effect.⁶⁾ In particular, the deflection angle is written as $4GM/bc^2$, where G is the gravitational constant, M is the lens mass, b is the impact parameter of the light ray, and c is the speed of light. For the derivation of the deflection angle and consequently the lens equation, the post-Minkowskian approximation $O(G)$ or the linear-order approximation $O(M)$ of the Schwarzschild metric is employed.

Do second-order relativistic corrections affect microlensing through the amplification factor? This has been already answered.⁷⁾ The total amplification remains unchanged at $O(G^2M^2)$, whereas the magnification factor for each image is corrected at this order.

The main purpose of this paper is to generalize their relativistic result. In particular, theoretical models beyond the theory of general relativity have attracted a lot of interests last decades, mostly motivated by the dark energy and dark matter problems. In this letter, therefore, we shall reexamine the amplification of lensed

images by taking account of corrections in a rather general form, where we assume static, spherically symmetric spacetimes.

We use the units of $c = 1$ but keep G in order to make iterative calculations clearer.

§2. Corrections to the lens equation and magnification

2.1. Standard form of lens equation and magnification

Let us begin by briefly summarizing the derivation of the total amplification for microlensing.¹⁾ We denote distances from the observer to the lens, from the observer to the source, and from the lens to the source as D_L , D_S and D_{LS} , respectively. The source and image position angles are denoted by θ_S and θ_I , respectively. The lens equation is written as

$$\theta_S = \theta_I - \frac{D_{LS}}{D_S} \alpha, \quad (1)$$

where we used the thin-lens approximation. This gives us a mapping between the lens and source planes. We can choose $\theta_S \geq 0$ without loss of generality. For a point mass lens (generally a spherically symmetric lens), the deflection angle α at $O(G)$ becomes

$$\alpha = \frac{4GM}{b}. \quad (2)$$

Hence, the lens equation is

$$\theta_S = \theta_I - \frac{\theta_E^2}{\theta_I}, \quad (3)$$

where θ_E is the angular radius of the Einstein ring as

$$\theta_E = \sqrt{\frac{4GM D_{LS}}{D_L D_S}}. \quad (4)$$

For microlensing in our galaxy, the radius is typically

$$\theta_E \sim 10^{-3} \left(\frac{M}{M_\odot} \right)^{1/2} \left(\frac{10 \text{ kpc}}{D_S} \right)^{1/2} \text{ arcsec.}, \quad (5)$$

where we assumed $D_L \sim D_{LS}$.

The lens equation is solved easily as

$$\theta_{\pm} = \frac{1}{2}(\theta_S \pm \sqrt{\theta_S^2 + 4\theta_E^2}). \quad (6)$$

We obtain the amplification due to gravitational lensing as

$$A = \left| \frac{\theta_I}{\theta_S} \frac{d\theta_I}{d\theta_S} \right| = \frac{1}{\left| 1 - \left(\frac{\theta_E}{\theta_I} \right)^4 \right|}, \quad (7)$$

which is a function of the image position θ_I .

For each image at θ_{\pm} , the amplification factor becomes

$$A_{\pm} = \frac{u^2 + 2}{2u\sqrt{u^2 + 4}} \pm \frac{1}{2}, \quad (8)$$

where u denotes θ_S/θ_E , the source position in units of the Einstein ring radius. For microlensing events, the angular separation between the images is too small to measure. All we can measure is the total amplification that is expressed as

$$A_{\text{total}} = A_+ + A_- = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}. \quad (9)$$

2.2. Modified bending angle and its effects on magnification

Let us take account of some modification in the bending angle of light rays. We focus on modified gravity theories that consider spacetimes to be *differentiable* manifolds. This implies that the modified bending angle can be expressed in terms of differentiable functions. We assume also that the lens object produces a static and spherically symmetric spacetime, for which the bending angle may take a general form as

$$\alpha = \frac{4GM}{b} [1 + F(b)], \quad (10)$$

with a differentiable function $F(b)$ denoting modifications. This point is contrast to previous works,^{7)–9)} in which particular forms such as the second-order of mass in the Schwarzschild spacetime and the PPN formalism are assumed.

For specific models, $F(b)$ may be approximated in terms of power functions b^p .¹⁰⁾ For instance, $p = 1/2$ for the DGP model as one of brane scenarios,¹¹⁾ and $p = 3/2$ for a candidate of massive graviton theories.^{12), 13)}

The lens equation to be solved becomes

$$\tilde{\theta}_S = \tilde{\theta}_I - \tilde{\alpha}(\tilde{\theta}_I), \quad (11)$$

with the modified bending angle as

$$\tilde{\alpha}(\tilde{\theta}_I) \equiv \frac{1}{\tilde{\theta}_I} \left(1 + \varepsilon f(|\tilde{\theta}_I|) \right), \quad (12)$$

where we introduce a nondimensional small quantity ε as the expansion parameter and $f(|\tilde{\theta}_I|)$ is an arbitrary differentiable function corresponding to $F(b)$. Here, $\tilde{\theta}_S$ and $\tilde{\theta}_I$ are normalized by θ_E as $\tilde{\theta}_S \equiv \theta_S/\theta_E = u$ and $\tilde{\theta}_I \equiv \theta_I/\theta_E$, respectively. We should note that $f(\tilde{\theta}_I = +\infty)$ can be absorbed into the mass that is defined at the spatial infinity and hence $f(\tilde{\theta}_I = +\infty)$ can be taken as zero. Henceforth, we assume $f(\tilde{\theta}_I) \rightarrow 0$ as $\tilde{\theta}_I \rightarrow \infty$.

This lens equation is rewritten as

$$\tilde{\theta}_I^2 - \tilde{\theta}_S \tilde{\theta}_I - 1 - \varepsilon f(|\tilde{\theta}_I|) = 0. \quad (13)$$

The lens equation (13) can be solved iteratively in terms of ε . For $\tilde{\theta}_I \geq 0$, the lens equation is rewritten as

$$\tilde{\theta}_I^2 - \tilde{\theta}_S \tilde{\theta}_I - 1 - \varepsilon f(\tilde{\theta}_I) = 0. \quad (14)$$

A perturbative form as

$$\tilde{\theta}'_+ = \tilde{\theta}_+ + \varepsilon \phi_+ + O(\varepsilon^2), \quad (15)$$

is substituted into the image position $\tilde{\theta}_I$ in Eq. (14), where the prime denotes a quantity including effects by the modified bending angle. Here, one can assume $\tilde{\theta}'_+ > 0$ because of $\tilde{\theta}_+ > 0$, within the limit that perturbative calculations hold. Then, we find the solution as

$$\tilde{\theta}'_+ = \tilde{\theta}_+ + \varepsilon \frac{f(\tilde{\theta}_+)}{\sqrt{\tilde{\theta}_S^2 + 4}} + O(\varepsilon^2). \quad (16)$$

For $\tilde{\theta}_I < 0$, the lens equation becomes

$$\tilde{\theta}_I^2 - \tilde{\theta}_S \tilde{\theta}_I - 1 - \varepsilon f(-\tilde{\theta}_I) = 0. \quad (17)$$

In the similar manner to the positive case, one finds the solution as

$$\tilde{\theta}'_- = \tilde{\theta}_- - \varepsilon \frac{f(-\tilde{\theta}_-)}{\sqrt{\tilde{\theta}_S^2 + 4}} + O(\varepsilon^2). \quad (18)$$

By using Eqs. (16) and (18), we obtain the amplification for each image as

$$A'_\pm = \frac{u^2 + 2}{2u\sqrt{u^2 + 4}} \pm \left(\frac{1}{2} + \varepsilon \ell_\pm \right) + O(\varepsilon^2), \quad (19)$$

where we define the correction term ℓ_\pm as

$$\ell_\pm \equiv \pm \frac{\tilde{\theta}_\pm}{\tilde{\theta}_S} \frac{d\tilde{\theta}_\pm}{d\tilde{\theta}_S} \frac{1}{\sqrt{u^2 + 4}} \left(\frac{f(\pm\tilde{\theta}_\pm)}{\tilde{\theta}_\pm} + \frac{df(\pm\tilde{\theta}_\pm)}{d\tilde{\theta}_\pm} \right) \mp f(\pm\tilde{\theta}_\pm) \frac{\tilde{\theta}_S}{(u^2 + 4)^{3/2}}. \quad (20)$$

Here, we assume that the sign of the Jacobian for the lens mapping does not change at the linear order of the modified bending angle as far as perturbative treatments are valid. Clearly the modified bending angle leads to a change in the amplification for each image.

By using Eq. (19), we obtain the total amplification as

$$A'_{\text{total}} = \frac{u^2 + 2}{u\sqrt{u^2 + 4}} + \varepsilon(\ell_+ - \ell_-) + O(\varepsilon^2). \quad (21)$$

According to Eq. (21), $\ell_+ = \ell_-$ is equivalent to the condition that the total amplification for the modified bending angle remains unchanged at the linear order. By using Eq. (20), this condition is rewritten as

$$\frac{d}{du} \frac{\tilde{\theta}_+ f(\tilde{\theta}_+)}{\sqrt{u^2 + 4}} = \frac{d}{du} \frac{(-\tilde{\theta}_-) f(-\tilde{\theta}_-)}{\sqrt{u^2 + 4}}, \quad (22)$$

where we should note that $\tilde{\theta}_\pm$ is a function of u . This is integrated as

$$\frac{\tilde{\theta}_+ f(\tilde{\theta}_+) + \tilde{\theta}_- f(-\tilde{\theta}_-)}{\sqrt{u^2 + 4}} = \text{const.} \quad (23)$$

For the asymptotically flat case, this constant must vanish as shown below. In the limit as $u \rightarrow \infty$, we have $\tilde{\theta}_+ \rightarrow u \rightarrow \infty$ and $\tilde{\theta}_- \rightarrow 0$. Hence the L.H.S. of Eq. (23) becomes $f(\tilde{\theta}_+ = \infty) = 0$ as $u \rightarrow \infty$, which means that the above constant vanishes. Therefore, the condition of $\ell_+ = \ell_-$ is expressed as

$$\tilde{\theta}_+ f(\tilde{\theta}_+) = -\tilde{\theta}_- f(-\tilde{\theta}_-). \quad (24)$$

By using the identity as $\tilde{\theta}_+ \tilde{\theta}_- = -1$, this is rewritten simply as

$$g(x) = g\left(\frac{1}{x}\right), \quad (25)$$

where we define $g(x) \equiv x f(x)$. Let us prove that $g(x)$ is a constant. We expand the differentiable function $g(x)$ in the Laurent series as

$$g(x) = \sum_{r=-\infty}^{\infty} a_r x^r, \quad (26)$$

where a_r is some constant and singular points may exist. This gives also $g(1/x)$ as

$$g\left(\frac{1}{x}\right) = \sum_{r=-\infty}^{\infty} a_{-r} x^r. \quad (27)$$

For these expansions, Eq. (25) tells us

$$a_r = a_{-r}, \quad (28)$$

which allows us to rearrange the expansion of $g(x)$ as

$$g(x) = a_0 + a_1 \left(x + \frac{1}{x}\right) + a_2 \left(x^2 + \frac{1}{x^2}\right) + \cdots. \quad (29)$$

The asymptotic flatness requires that the bending angle vanishes at the spatial infinity. Namely, $x^{-1} f(x)$ vanishes as $x \rightarrow \infty$, which means $a_r = 0$ for $r \geq 2$. Then, $g(x) = a_0 + a_1(x + x^{-1})$ gives the bending angle defined by Eq. (12) as

$$\tilde{\alpha}(\tilde{b}) = \frac{1}{\tilde{b}} \left[1 + \varepsilon \left\{ \frac{a_0}{\tilde{b}} + a_1 \left(1 + \frac{1}{\tilde{b}^2} \right) \right\} \right]. \quad (30)$$

However, the two terms with a_1 are rewritten as

$$1 + \frac{\theta_E^2}{\theta_I^2} = 1 + \frac{4GM D_L D_{LS}}{D_S b^2}. \quad (31)$$

This expression makes no sense, because the bending angle is calculated by assuming the observer and source located at the null infinity and hence it includes neither D_{LS} , D_L nor D_S . Therefore, a_1 must vanish. It should be noted that this subtle argument is not true of Eq. (26), since there are a lot of ways for combining terms in an infinite series.

As a result, we find only $g(x) = a_0$, which leads to $f(x) = a_0/x$. This equation states exactly that the modified bending angle is proportional to the inverse square of the impact parameter (e.g., in Eq. (12)). One example of such inverse-square corrections is the second-order approximation of the Schwarzschild lens.^{7),14)} Another example in general relativity is the Reissner-Nordstrom solution representing a charged black hole, for which the correction to the bending angle is $\sim q^2/b^2$ for a small charge q .

Note that there is a subtle but large difference between treatments of a_1 and a_0 . This is because the deflection angle depends on the lens mass and impact parameter but not any other distances such as D_{LS} in the lensing theory based on the null infinity condition.⁶⁾ The coefficient of a_1 is given by Eq. (31) as the *ill-defined* sum of the (nondimensional) unity and the combination of the mass and distances, so that the distances cannot be removed by using the impact parameter. On the other hand, a_0 has a coefficient as the inverse of the impact parameter squared. Hence, the coefficient is well-defined and thus allowed.

2.3. Discussion

Do modified magnification factors which are discussed above have any implications for observations? In microlensing, u has a definite dependence on time t , which is expressed as $u = (u_0^2 + v_T^2 t^2)^{1/2}$ for u_0 corresponding to the minimum of the impact parameter and v_T denoting the transverse angular velocity of the source with respect to the lens. The modified total amplification thus has a different dependence on time and thus produces a different shape of the light curve. It should be noted that the peak height of the light curve is *less* informative on modifications, because it depends on many parameters and hence there is a degeneracy in parameter spaces. Shapes of light curves are of greater importance. In principle, accurate measurements of light-curve shapes can be used for detecting (or observationally constraining) the modifications discussed above.

For instance, we take a correction to the deflection angle as the Yukawa-type function as

$$\frac{F(b)}{b} = \frac{e^{-Kb}}{b}, \quad (32)$$

where K is roughly the inverse of the characteristic length in a modified gravity theory or the inverse of a new mass scale. Figures 1 and 2 show effects of the modified bending angle on light curves. Figure 1 gives shapes of light curves for two cases. One is based on the amplification factor in the standard form and the other is for the Yukawa-type modification as $f(\tilde{b}) = \exp(-\tilde{K}\tilde{b})$. In order to make it clearer the deviation from the standard shape of the light curve, we define the ratio as

$$\delta A(u) \equiv \frac{A'_{\text{total}} - A_{\text{total}}}{A_{\text{total}}}, \quad (33)$$

where A_{total} and A'_{total} denote the standard form of total amplification and the modified one, respectively. Figure 2 plots δA , the deviation from the standard shape of the light curve. For these figures, we choose $u_0 = 1$, $\varepsilon = 0.2$ and $\tilde{K} = 1$ so that effects by the modification can be distinguished by eye.

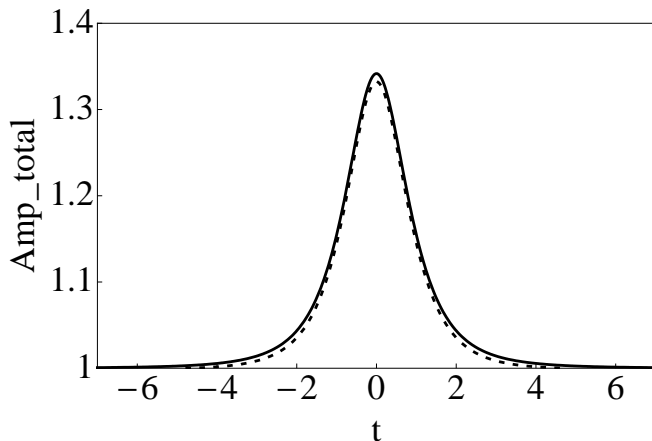


Fig. 1. Light curves for $u_0 = 1$: The solid curve is based on the total amplification in the standard form. The dashed one includes the Yukawa-type correction with $\varepsilon = 0.2$ and $\tilde{K} = 1$. Time t is in units of the Einstein cross time as θ_E/v_T .

For this model, α is *effectively* increased and hence A_{total} seems to be enhanced. However, A_{total} is decreased as shown by Fig. 2. This is because the amplification is caused by not the bending angle but its *derivative*. In fact, the Yukawa-type correction gives $df(\tilde{b})/d\tilde{b} = -\tilde{K} \exp(-\tilde{K}\tilde{b}) < 0$, namely a minus contribution to the amplification factor. So far, no systematic deviation in microlensing observations has been reported. One can thus put a constraint on K as $K^{-1} > 1000 \text{ AU} \sim 10^{14} \text{ m}$ for $\varepsilon \sim 1$, where we assume the photometric accuracy comparable to 0.1 percents and $b \sim D_L \theta_E \sim O(1 \text{ AU})$. This bound is consistent with gravitational experiments in the solar system.

Finally, we mention an asymptotically massless spacetime, for which the above theorem still stands by taking $M \rightarrow 0$ and considering the linear order in ε . One example is Ellis' wormhole that makes a difference in light curves,¹⁵⁾ since the bending angle for this spacetime is $\sim b^{-2}$ at the leading order but $\sim b^{-4}$ at the next order.

§3. Conclusion

We investigated corrections to the conventional lens equation in terms of differentiable functions, so that they can express not only the second-order effects of GM in general relativity but also modified gravity theories. It was shown that, provided that the spacetime is static, spherically symmetric and asymptotically flat, the total amplification by microlensing remains unchanged at the linear order of the correction to the deflection angle, if and only if the correction takes a particular form as the inverse square of the impact parameter, whereas the magnification factor for each image is corrected. It is concluded that the light curve shape by microlensing is inevitably changed and will thus allow us to probe modified gravity, unless a modification to the deflection angle takes the particular form.

It is left as a future work to use the present formulation to probe modified

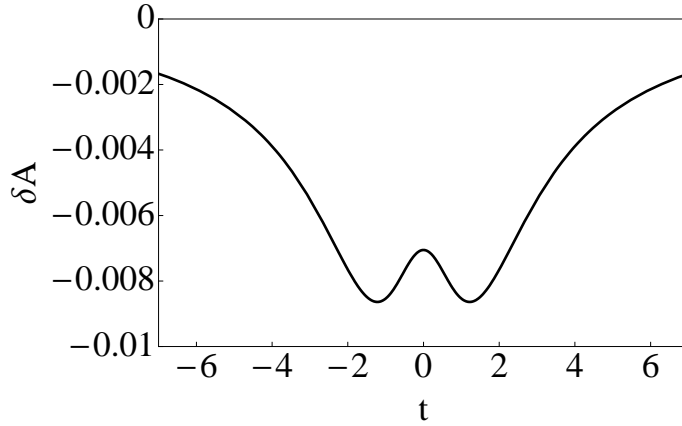


Fig. 2. The difference of the two light curves in Figure 1: This shows the dependence of δA on time (through the impact parameter).

gravity models by microlensing observations. Microlensing in our galaxy is sensitive to gravity at short scale around a few AU, whereas cosmological microlensing is more useful for the large distance physics.⁶⁾

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